Prove that $0 < a < 1 \implies$

$$1 > a > a^2 > a^3 > \dots$$

from begining facts, we have that

$$\boxed{\begin{array}{l} 3)\ a < b \text{ and } c > 0 \\ \qquad \implies ac < bc \end{array}}$$

applying this to this problem

$$a > 0 \quad \text{and} \quad a < 1$$

$$\Downarrow$$

$$a \cdot a < a \cdot 1$$

$$a^2 < a$$

$$\Downarrow$$

$$a^3 < a^2$$

$$\vdots$$

$\Rightarrow \quad 1 > a > a^2 > a^3 > \ldots$

Exercise 5.1.6

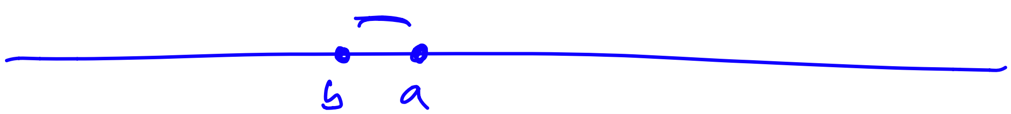
Suppose $a \le b + \varepsilon \quad \forall \varepsilon > 0$

prove that $a \le b$

Hint: Proof by contradiction

suppose not.... i.e. suppose

$$a > b$$



5.1.2

Banach Fixed Point theorem

$$(\underset{\substack{\uparrow \\ \text{liner space}}}{B}, \| \ \|)$$

$$F : B \to B$$

$$\underline{|F(x) - F(y)| < k|x - y|}, \quad 0 < k < 1$$

$$\Rightarrow \exists \ x^* \ni F(x^*) = x^*$$

x_0

$\bullet \ x_0$

$\bullet \ x_1 = F(x_0)$

$\bullet \ x_2 = F(x_1)$

$$x_{i+1} = F(x_i)$$

$$|\underset{\substack{\uparrow \\ F(x_i)}}{x_{i+1}} - \underset{\substack{\uparrow \\ F(x_{i-1})}}{x_i}| \le k|x_i - x_{i-1}|$$

1

$x_i \qquad x_j$

$$|x_{i+m} - x_i| \le |x_{i+m} - x_{i+m-1}| + |x_{i+m-1} - x_{i+m-2}|$$
$$+ |x_{i+m-2} - x_{i+m-3}|$$
$$\dots + |x_{i+1} - x_i|$$

$$|x_2 - x_1| \le \underline{K|x_1 - x_0|}$$
$$|x_3 - x_2| \le \underline{K\,|x_2 - x_1|}$$
$$|x_4 - x_3| \le \underline{K\,|x_3 - x_2|} < K^2|x_2 - x_1|$$
$$< K^3|x_1 - x_0|$$

$$|x_{m+1} - x_m| \le K^{m}|x_1 - x_0|$$

$$|x_{i+m} - x_i| \le |x_{i+m} - x_{i+m-1}| + |x_{i+m-1} - x_{i+m-2}|$$
$$+ |x_{i+m-2} - x_{i+m-3}|$$
$$\dots + \underline{|x_{i+1} - x_i|}$$

$$|x_{i+m} - x_i| \le k^i |x_1 - x_0| + k^{i+1} |x_1 - x_0| + k^{i+2} |x_1 - x_0| + \cdots k^{i+m-1} |x_1 - x_0|$$

$$= k^i \underbrace{(1 + k + k^2 + \cdots + k^{m-1})}_{\frac{1}{1-k}} |x_1 - x_0|$$

$$\le \frac{k^i}{1-k} |x_1 - x_0|$$

$$\lim_{i \to \infty} F(x_i) = F(\underbrace{\lim_{i \to \infty} x_i}_{x^*}) = F(x^*)$$

$$\lim_{i \to \infty} x_{i+1} = x^*$$

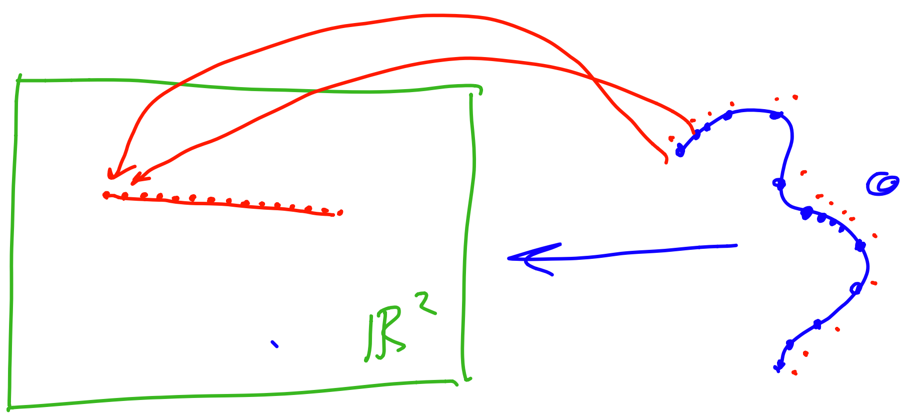
$$F(x^*) = x^*$$

Inclusion vs Embedding

just set inclusion

$A \subset B$ means all points in A are in B.

embedding is used when we want to say one set (often of functions) is a subspace of another.



$$\dot{x} = f(x)$$

$$\Downarrow$$

$$x_1(t) \equiv \underbrace{\int_0^t f(x_0(s))\,ds + x_0(0)}_{F(x_0)}$$

more generally

$$x_i(t) \equiv \int_0^t f(x_{i-1}(s))\,ds + x_{i-1}(0)$$

$\hat{x}$
a solution

$$\hat{x}(t) = \int_0^t f(\hat{x}(s))\,ds + \hat{x}(0)$$

$$\dot{\hat{x}} = f(\hat{x})$$

$\downarrow$ integrate

$$\hat{x}(t) - \hat{x}(0) = \int_0^t \dot{\hat{x}}(s)\,ds = \int_0^t f(\hat{x}(s))\,ds$$

$$\underline{\hat{x}(t) = \underbrace{\int_0^t f(\hat{x}(s))\,ds + \hat{x}(0)}_{F(\hat{x}(s))}}$$

$$X_i(t) = \int_0^t \underline{f}(X_{i-1}(s))\,ds + X_{i-1}(0)$$