$\mathbb{R}^n$, $\|\cdot\|$

$$\|x\| \equiv \sqrt{\langle x, x \rangle}$$
$$= \sqrt{x \cdot x}$$
$$= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

---

$$M = (X, \rho)$$

triangle inequality

set of points

vector space

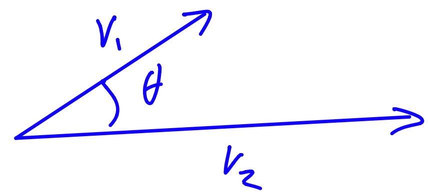
$$M = (V, \rho(x,y)), \quad \rho(x,y) \equiv \|x-y\| = \sqrt{\langle x-y, x-y \rangle}$$

because $X = V$, vector space

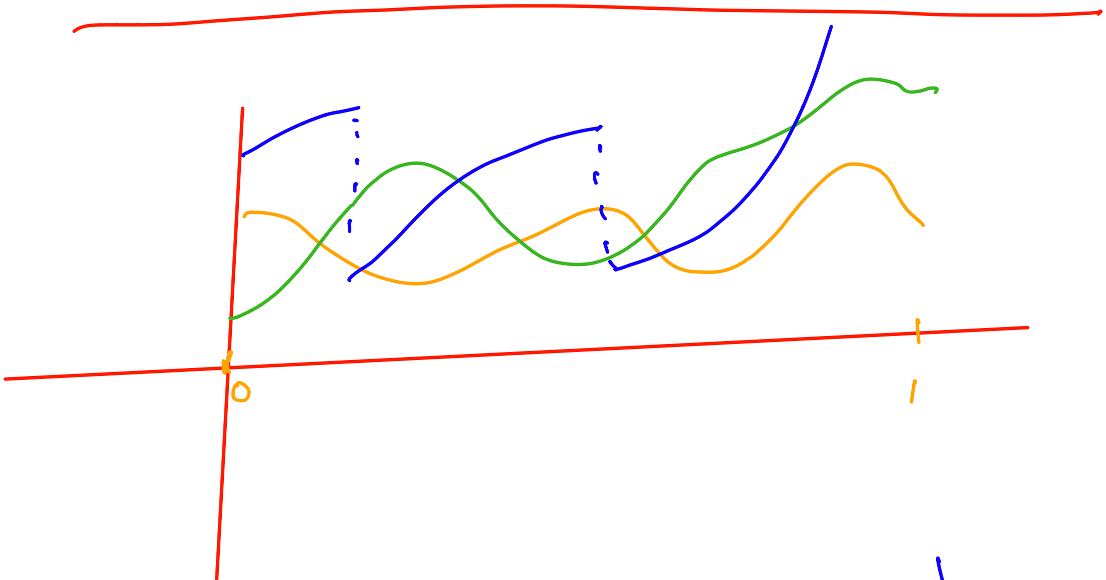
we can add & subtract points

because $\|\cdot\|$ comes from an inner product angles are defined

$\mathbb{R}^n$




$$\cos^{-1}\left(\frac{v_1 \cdot v_2}{\|v_1\|\,\|v_2\|}\right) = \theta$$




$\rightarrow$ $X \equiv$ space of functions

$f : [0,1] \rightarrow \mathbb{R}$

$\ni \displaystyle\int_0^1 |f|^2 \, dx < \infty$

$$\to \quad \|f\| \equiv \left( \int_0^1 |f|^2 \, dx \right)^{1/2}$$

$$\to \quad \langle f, g \rangle \equiv \int_0^1 f g \, dx$$

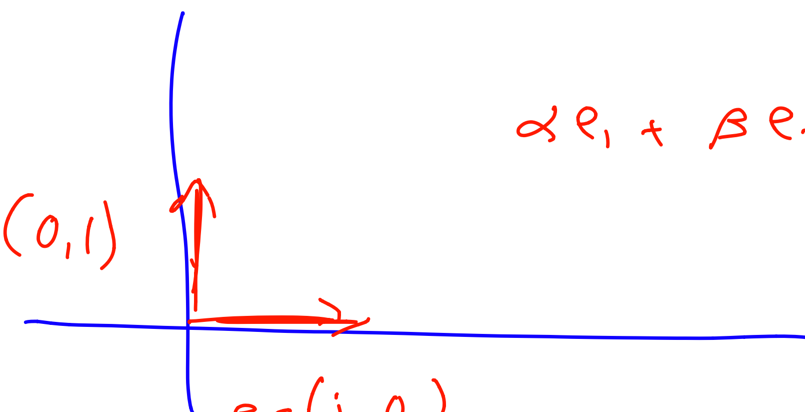
$$\Downarrow$$

$$\|f\| = \sqrt{\langle f, f \rangle}$$

$\mathcal{L}^2([0,1])$

$$\alpha e_1 + \beta e_2 = (\alpha, \beta)$$

$e_2 \equiv (0,1)$

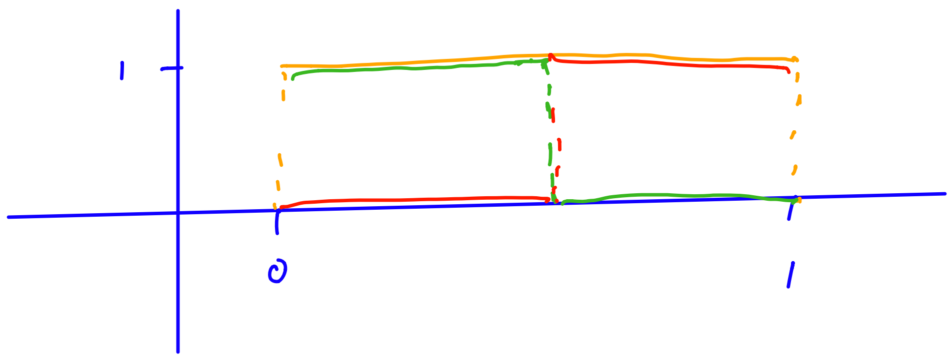


$e_1 \equiv (1,0)$

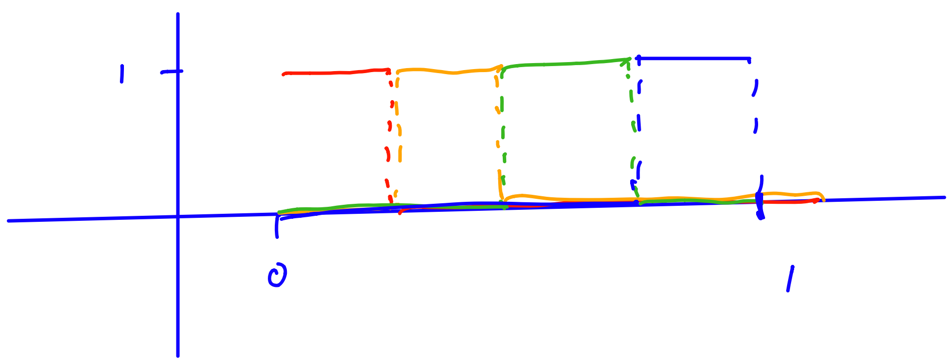
---

$$f \in \mathcal{L}^2([0,1])$$

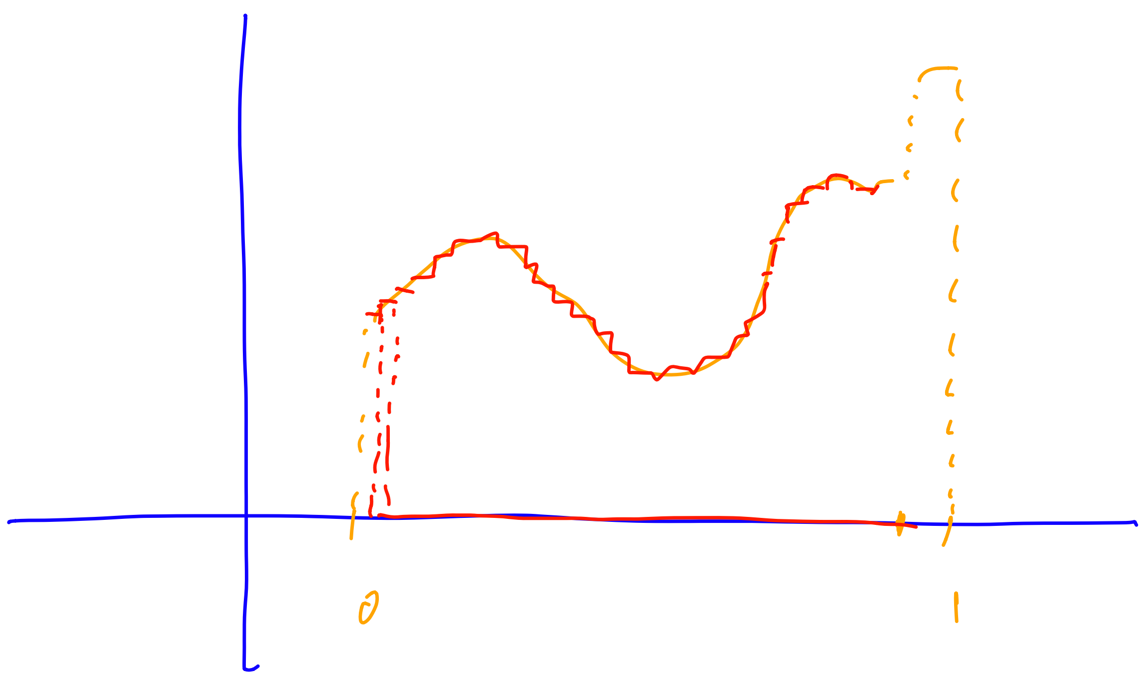
$$f = \sum_{i=1}^{\infty} \alpha_i h_i \quad , \quad h_i \in \mathcal{L}^2([0,1])$$



1
0
1

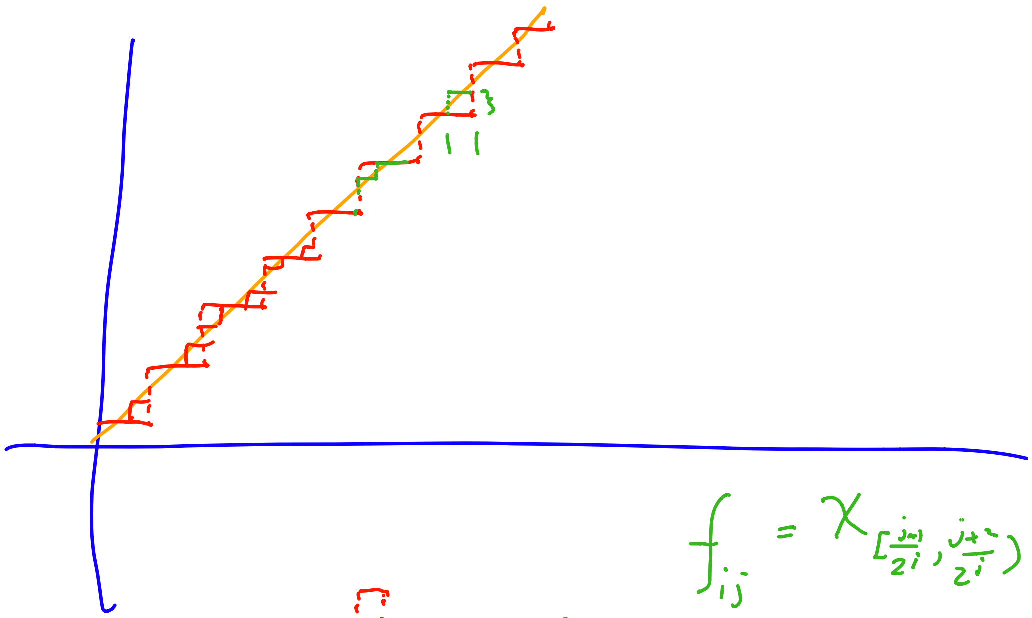



1
0
1

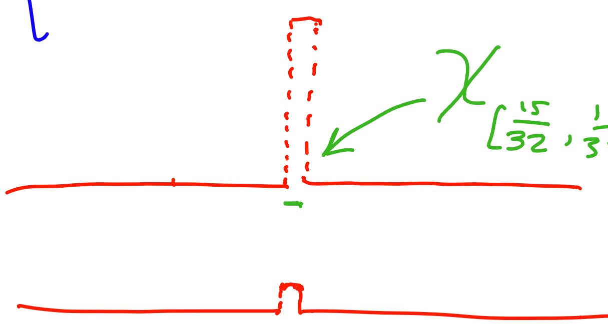



0
1

$\frac{1}{256}$



$f_{ij} = \chi_{\left[\frac{j+1}{2^i}, \frac{j+2}{2^i}\right)}$



$\chi_{\left[\frac{15}{32}, \frac{16}{32}\right)} = f_{5,14}$

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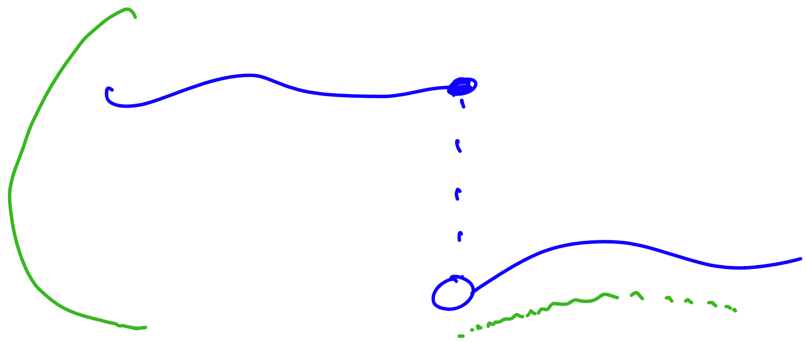
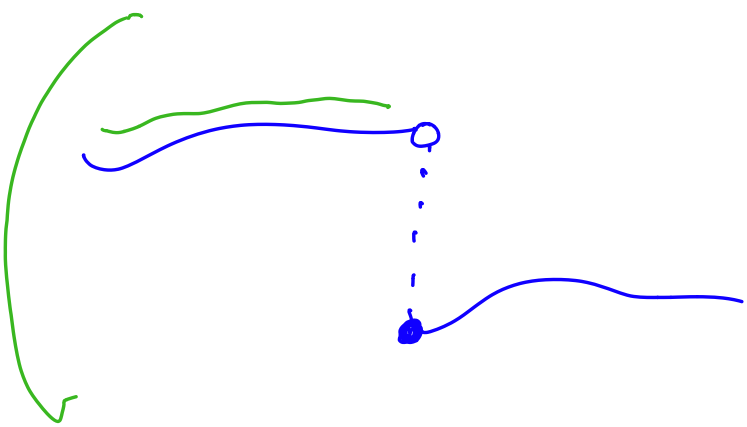
<u>Questions</u>:

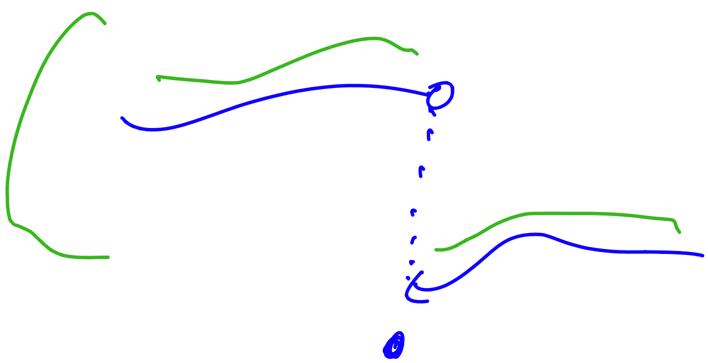
$\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^{10}, \ldots \mathbb{R}^\infty$?



1



2



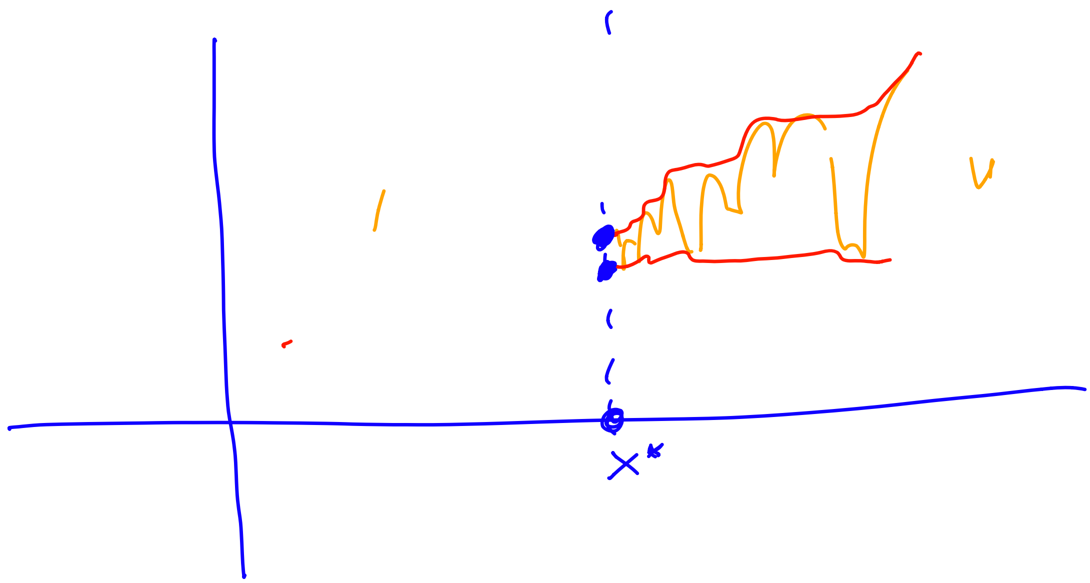
3

$f$ is upper semi-continuous at $x$

and

f is lower semi-continuous at x

then $\Longrightarrow$ f is continuous at x




$$f(x^*) \geq \limsup_{x \to x^*} f(x)$$

$$f(x^*) \leq \liminf_{x \to x^*} f(x)$$