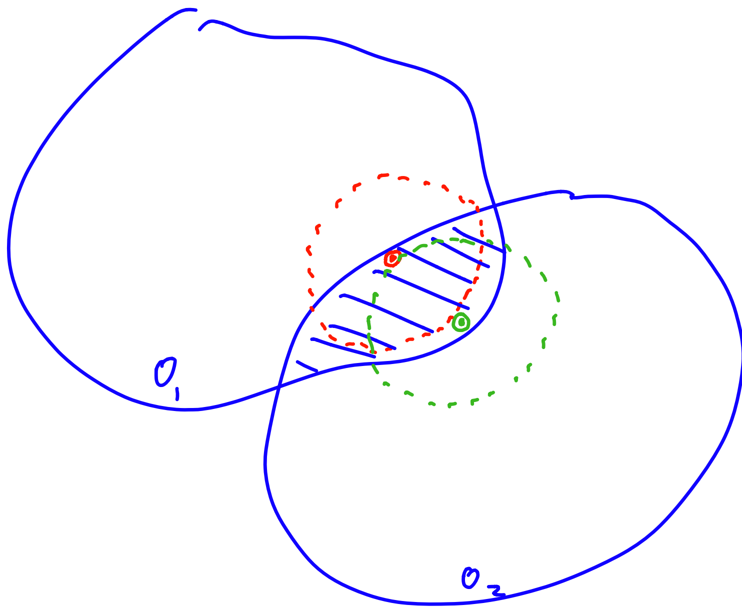
Exercise 4.2.2 Show that the intersection of a finite collection of open sets is open.

① $E \equiv \bigcap_{i=1}^{N} O_i$ $\qquad O_i$ are open

② choose $x \in E$

$$x \in O_1 \implies B(x, r_1) \subset O_1$$
$$x \in O_2 \implies B(x, r_2) \subset O_2$$
$$x \in O_3 \implies B(x, r_3) \subset O_3$$
$$\vdots$$
$$x \in O_N \implies B(x, r_N) \subset O_N$$



Example when $N=2$

③ order r_i's

example $0 < r_{10} \le r_1 \le r_N \le r_3 \le \dots$

$N = 4$

$r_1 = 1$

$r_2 = .3$

$r_3 = .5$

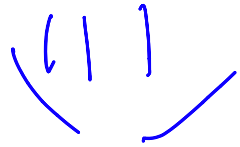
$r_4 = 1$

$0 < .3 < .5 < 1 \le 1$

$0 < r_2 \le r_3 \le r_1 \le r_4$

rename it r_{min}

$\Rightarrow \quad B(x, r_{min}) \subset B(x, r_i) \quad \forall i$

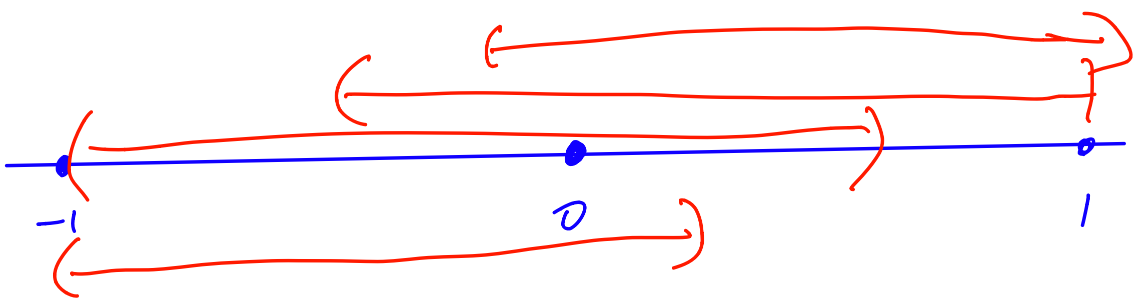


$$B(x, r_{min}) \subset O_i \quad \forall i \ !$$

$$\subset \bigcap_{i=1}^{N} O_i = E$$

$$\Longrightarrow E \text{ is open!}$$

Example



$(-1, \frac{1}{2})$

$(-\frac{1}{2}, 1)$

$(-1, \frac{1}{4})$

$(-\frac{1}{4}, 1)$

$\rightarrow (-\frac{1}{2}, \frac{1}{2})$

$(-\frac{1}{4}, \frac{1}{4})$

$\vdots$

$(-\frac{1}{8}, \frac{1}{8})$

$\left(-\frac{1}{16}, \frac{1}{16}\right)$

$\left(-\frac{1}{32}, \frac{1}{32}\right)$

$\left(-\frac{1}{64}, \frac{1}{64}\right)$

$\left(-\frac{1}{128}, \frac{1}{128}\right)$

$\vdots$

$\{0\}$

$\left(-1, \frac{1}{2^K}\right)$

$\left(-\frac{1}{2^K}, 1\right)$

$\varepsilon > 0$

$\Downarrow$

$\exists\ K \ni$

$\frac{1}{2^K} < \varepsilon$

$\varepsilon < 0$

$\varepsilon < -\frac{1}{2^K} < 0$

$$\bigcap_{K=1}^{\infty} \left(-\frac{1}{2^K}, \frac{1}{2^K}\right) = \{0\}$$

$$\overline{B}(x,r) = \{y \in X : \rho(x,y) \le r\}$$

$$B(x,r) = \{y \in X : \rho(x,y) < r\}$$

<u>Open set:</u> E is open $\iff \forall\, x \in E$

$\exists\ r > 0 \ni B(x,r) \subset E$

<u>closed set:</u> E is closed $\iff E^c$ is open

<u>Hausdorff space:</u>

$x \neq y \Rightarrow \exists\ O_x, O_y$ open

$x \in O_x$

$y \in O_y$

$$O_x \cap O_y = \emptyset$$