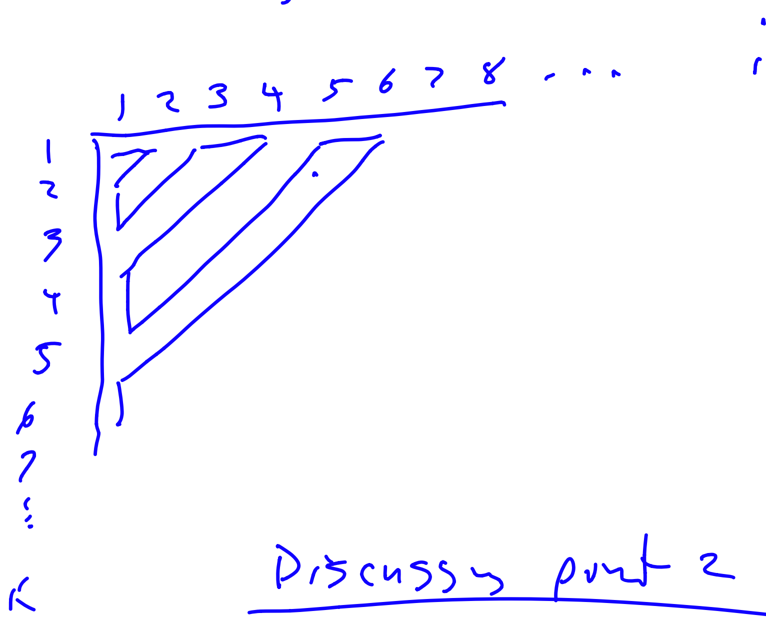
Show that in a separable metric space, X, every open cover has a countable subcover.

① $F \equiv \{f_i\}_{i=1}^{\infty} \subset X$ $\quad \forall\ x \in X$ and any $\varepsilon > 0$ $\quad \exists\ i \ni \rho(f_i, x) < \varepsilon$

$\Rightarrow$ $\exists$ sequences of points in F converging to any point in X

Discussion point 1

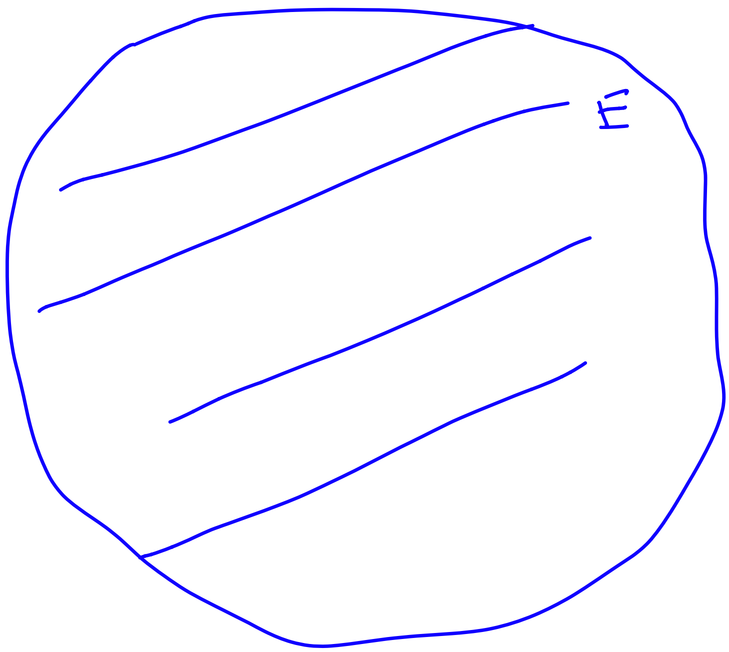
② define $\{B_{i,k}\} \equiv \{B(f_i, \frac{1}{2^k})\}$ Vrf



Discussion point 2

$$\Rightarrow \quad \{B_{i,k}\}_{i,k} = \{B_n\}_{n=1}^{\infty}$$

③ $\mathcal{O} = \{O_\alpha\}_{\alpha \in A}$



$$E \subset \bigcup_{\alpha \in A} O_\alpha$$

$\Downarrow$ goal

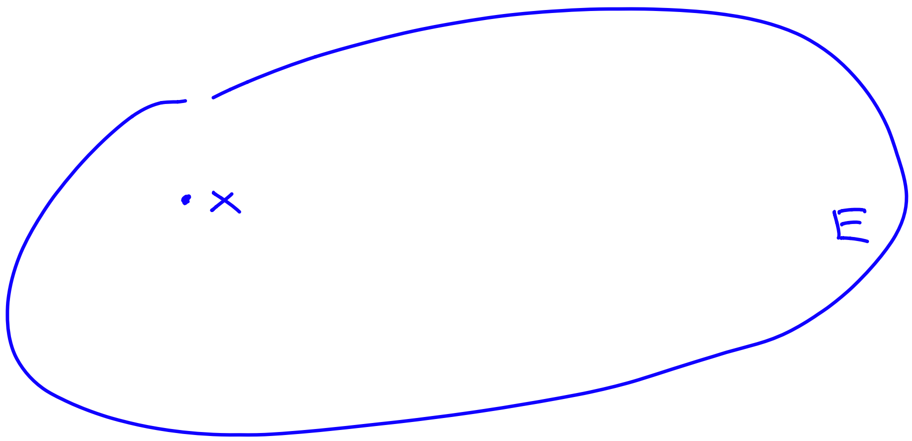
④ objective

$$\exists \{O_{\alpha_i}\}_{i=1}^{\infty} \ni E \subset \bigcup_{i=1}^{\infty} O_{\alpha_i}$$

Discussion point 3

5

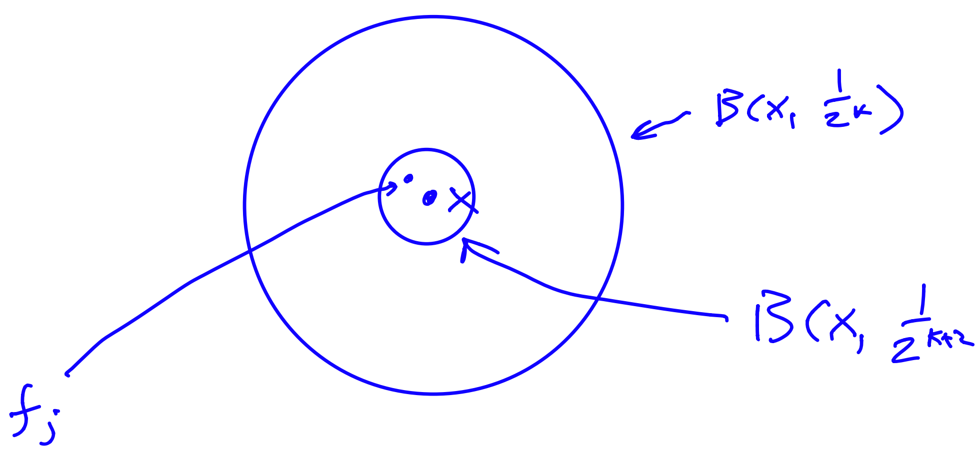
S1



S2 find $O_\alpha \ni x \in O_\alpha$

S3 because O_α is open $\exists\ B(x,r)$
$r > 0 \ni B(x,r) \subset O_\alpha$

S3 $\Rightarrow B(x, \frac{1}{2^k}) \subset O_\alpha$



claim: $x \in \underbrace{B\left(f_j, \frac{1}{2^{k+1}}\right)}_{B_\ell} \subset B\left(x, \frac{1}{2^{k}}\right) \subset O_\alpha$

point 1: Every $x \in E$ is in B_ℓ some ℓ

point 2: $\exists\, O_\alpha$ that we call O_{α_ℓ} containing B_ℓ

scan all $x \in E$

collect all B_ℓ's needed to contain all x's

then O_{α_ℓ}'s contain all x's

$$E \subset \bigcup_{\ell=1}^{\infty} O_{\alpha_\ell}$$