Defn I

$f: M_1 \to M_2$ is continuous if $f^{-1}(E)$ is open in M_1 when E is open in M_2

continuous f's pull open sets back to open sets

Reminder

$$f^{-1}(U) = \{x \in M_1 : f(x) \in U\}$$

Defn II

$\Rightarrow$ $f: M_1 \to M_2$ is continuous $\Longleftrightarrow$ if $x_i \to x^*$ then $f(x_i) \to f(x^*)$

Defn III

$f: M_1 \to M_2$ is continuous at x^*

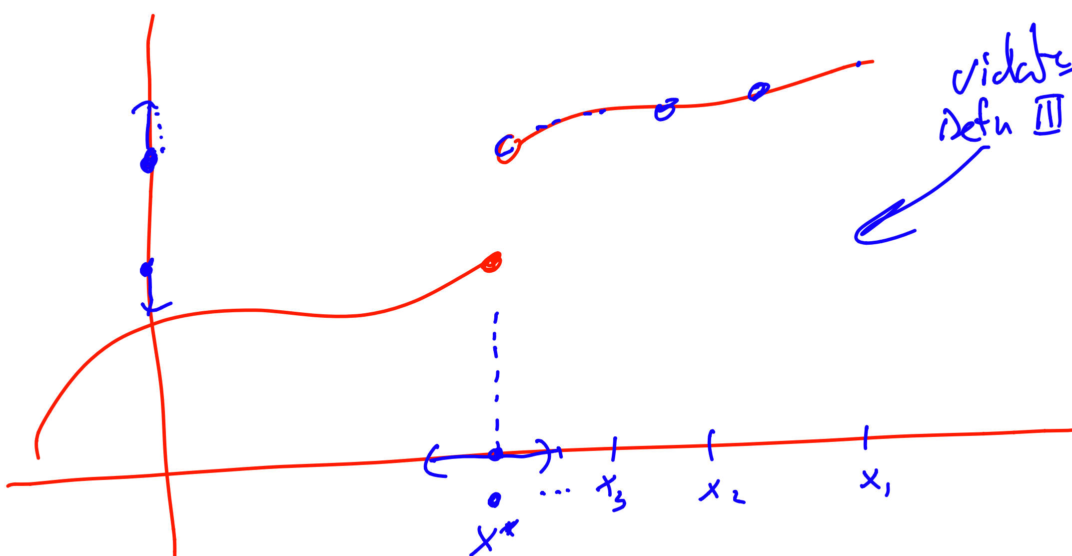
if for any $\varepsilon > 0$ $\exists$ $\delta(\varepsilon)$ $\ni$

$\rho_1(x^*, y) < \delta(\varepsilon) \Rightarrow \rho_2(f(x^*), f(y)) < \varepsilon$

if f is continuous at all $x \in M_1$

we say that f is continuous.

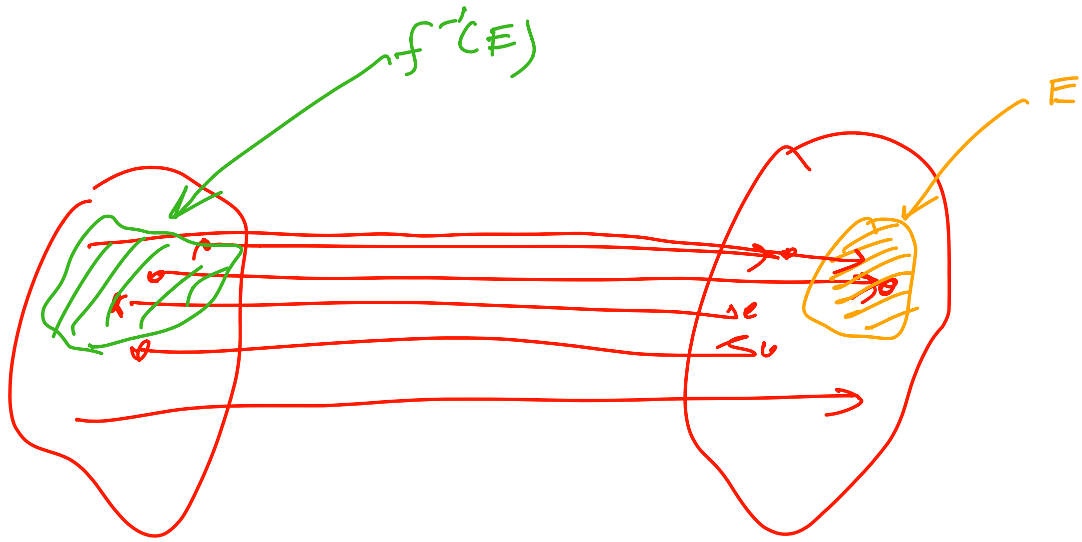
$$\boxed{B(x^*, \delta(\varepsilon)) \subset B(f(x^*), \varepsilon)}$$



violates Defn III

$x_i \to x^*$

$f(x_i)$ $f(x_i) \not\to f(x^*)$



Defn I $\Longrightarrow$ Defn II

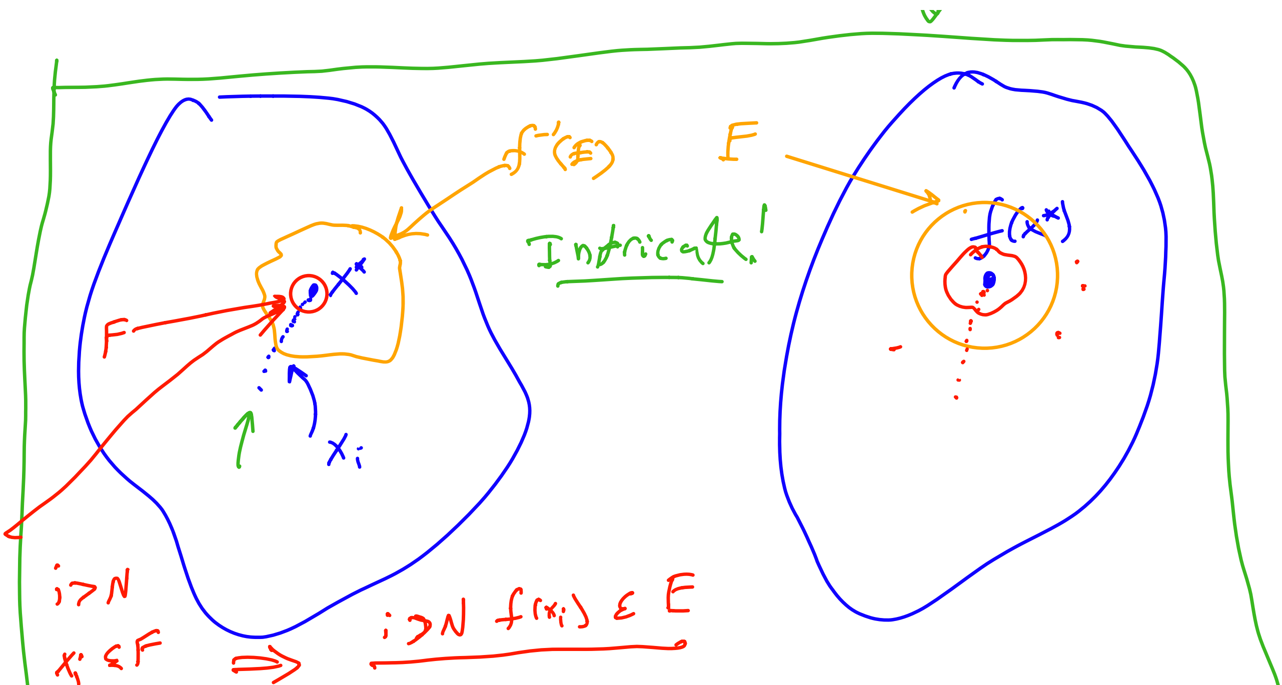
assume: $f^{-1}(E)$ open when E open (A)

want to prove: $x_i \to x^*$ $\Longrightarrow$ $f(x_i) \to f(x^*)$ (B)

assume A but not B ...

$\Longrightarrow$ $x_i \to x^*$ but

$f(x_i) \not\Rightarrow f(x^*)$?



$y_i \to y^*$: $\forall \varepsilon > 0 \ \exists N_\varepsilon \ni$

$i > N_\varepsilon \Rightarrow B(y^*, y_i) < \varepsilon$

$\Uparrow$ Showing such an f cannot exist!

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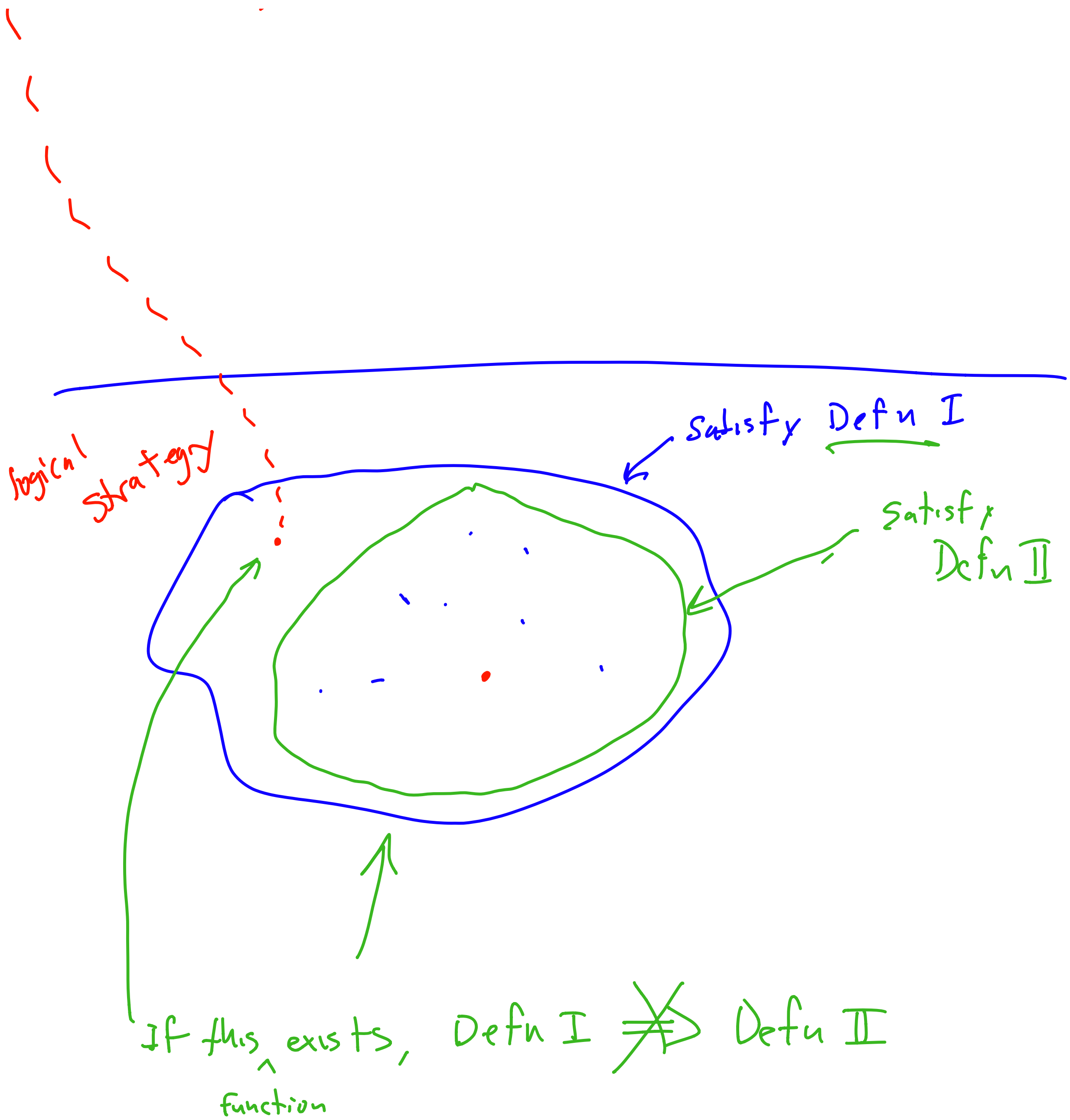
① $f^{-1}(E)$ is open when E open

② $\exists$ $x_i \to x^*$ and $f(x_i) \not\to f(x^*)$

contradicts Def II


logical strategy
Satisfy Defn I
Satisfy Defn II
If this exists, Defn I ⇏ Defn II
function