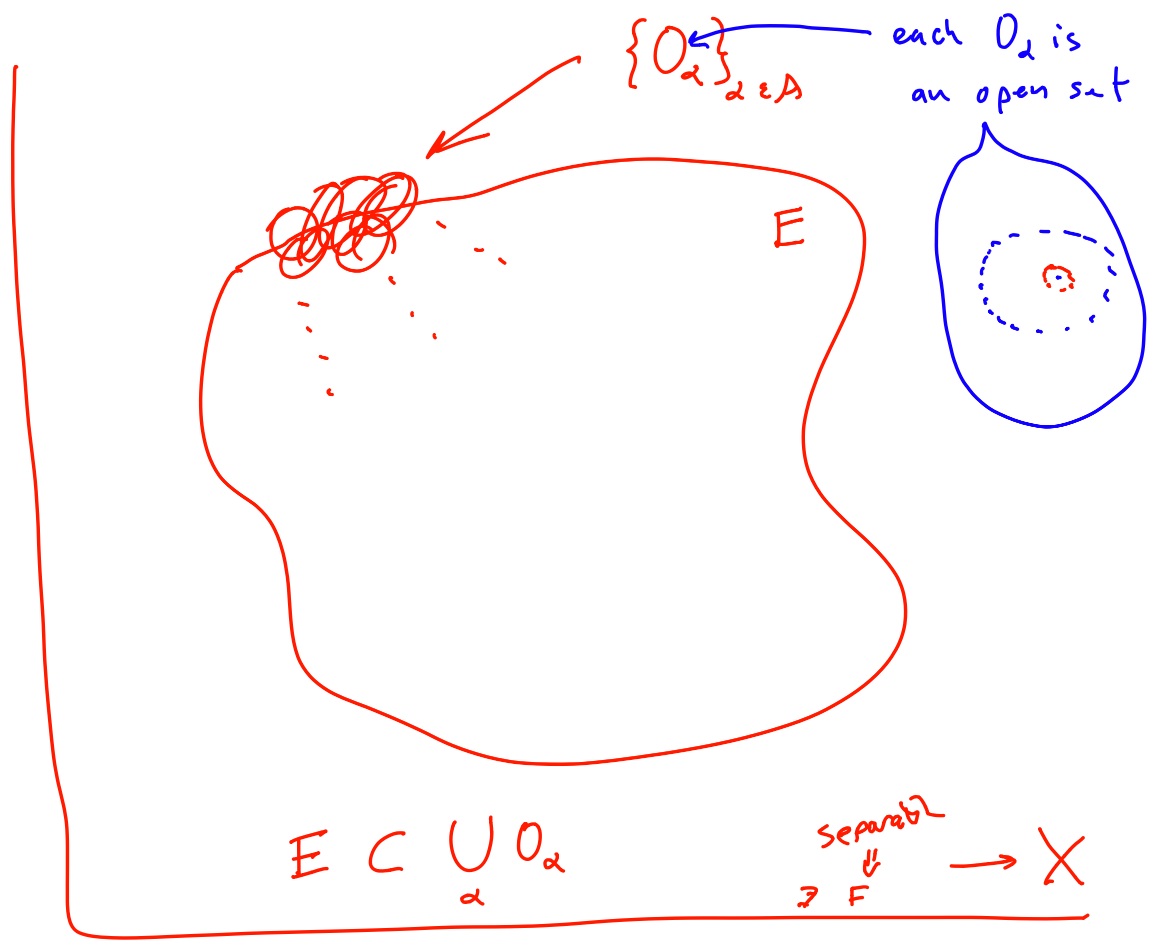
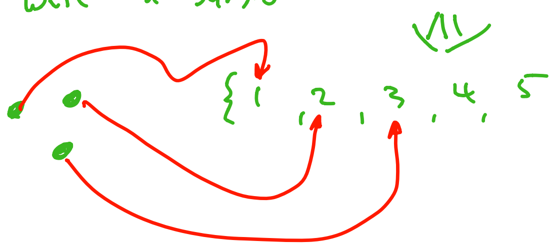


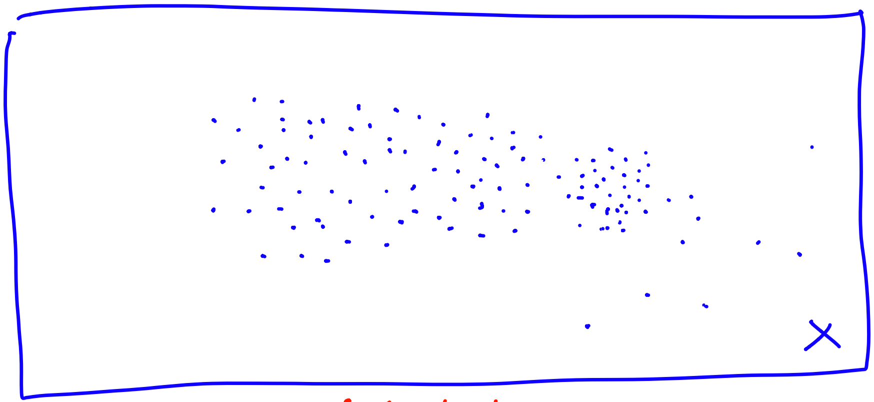

## Countable!

Something is "countable" if it can be put in 1-to-1 correspondence with a subset of $\mathbb{N}$




$$\{1, 2, 3, 4, 5, 6, \ldots\}$$

$$\bigcup_i \left\{\tfrac{1}{i}\right\} = \underbrace{\left\{\tfrac{1}{1}, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{6}, \tfrac{1}{7}, \ldots\right\}}_{\text{countably infinite subset of } \mathbb{R}}$$




$F = \{f_i\}_{i=1}^{\infty} \subset X$

$f_1 \quad f_2 \quad f_3 \quad f_4 \quad \ldots$

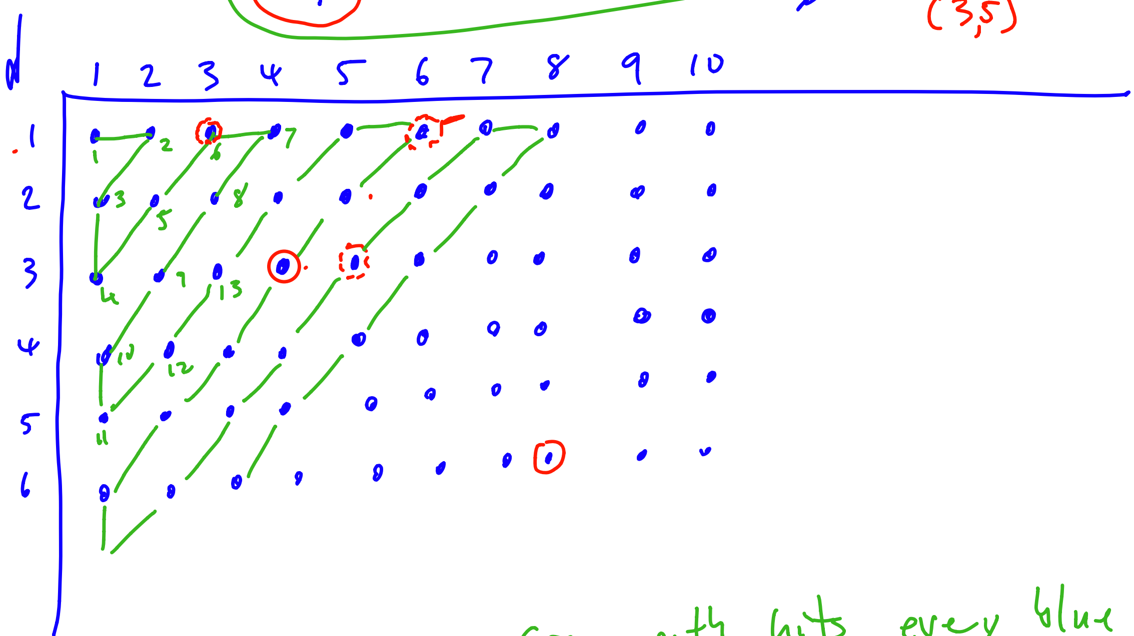
$1 \quad 2 \quad 3 \quad 4$

$B(f_3, \frac{1}{2^5})$

$B_{3,5}$

$(3,5)$

$B_{i,k} = B(f_i, \frac{1}{2^k})$




Green path hits every blue dot exactly once

$B_{1,3} \rightarrow B_6$

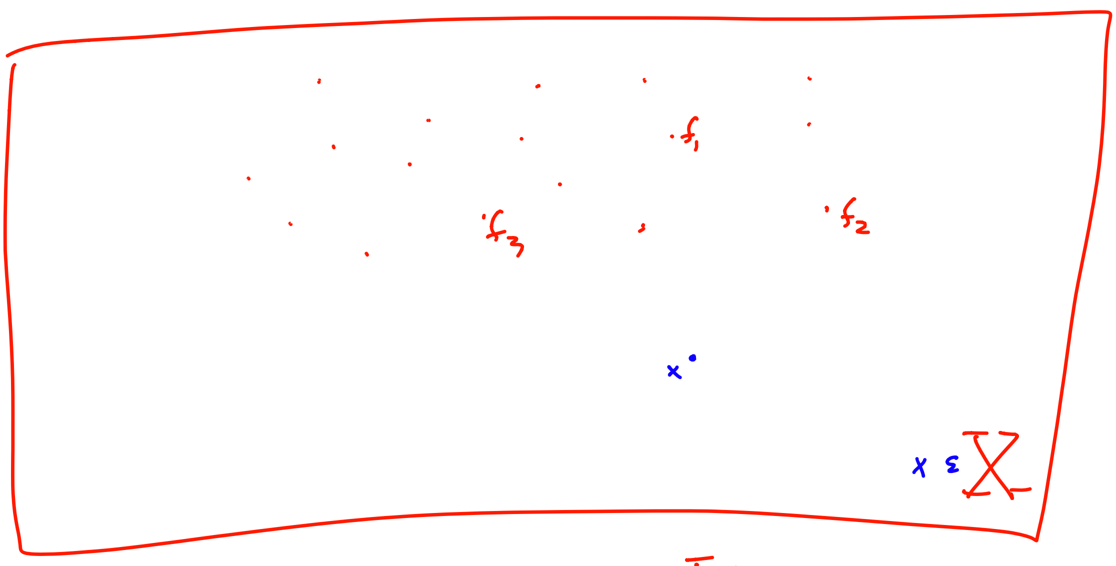
$B_{1,6} \rightarrow B_{16}$

$F$ dense $\iff$ $\forall\ x \in X$ and any $\varepsilon > 0$
$\exists\ f_j \in F$ so that

$$\boxed{x \in B(f_j, \varepsilon)} \iff \boxed{\rho(x, f_j) < \varepsilon}$$

equivalent

$X$ has $F$ with this property




set $\longrightarrow$ $F = \{f_i\}_{i=1}^{\infty} \subset X$

$$\exists\ j \ni \rho(f_j, x) < \frac{1}{2^{10^{100}}}$$

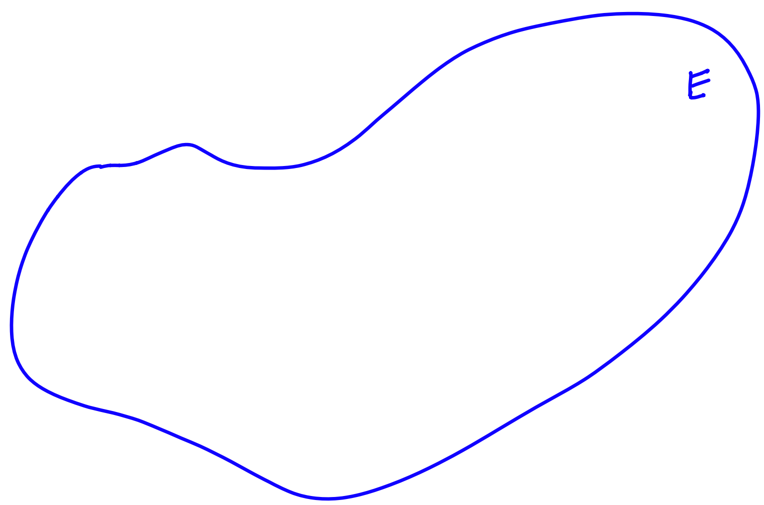
$F$ : understand separability

$B_{i,K}$ : construct special set of balls and realize I can relabel prove this set of balls is countable.

$\Downarrow$

$B_\ell$

---

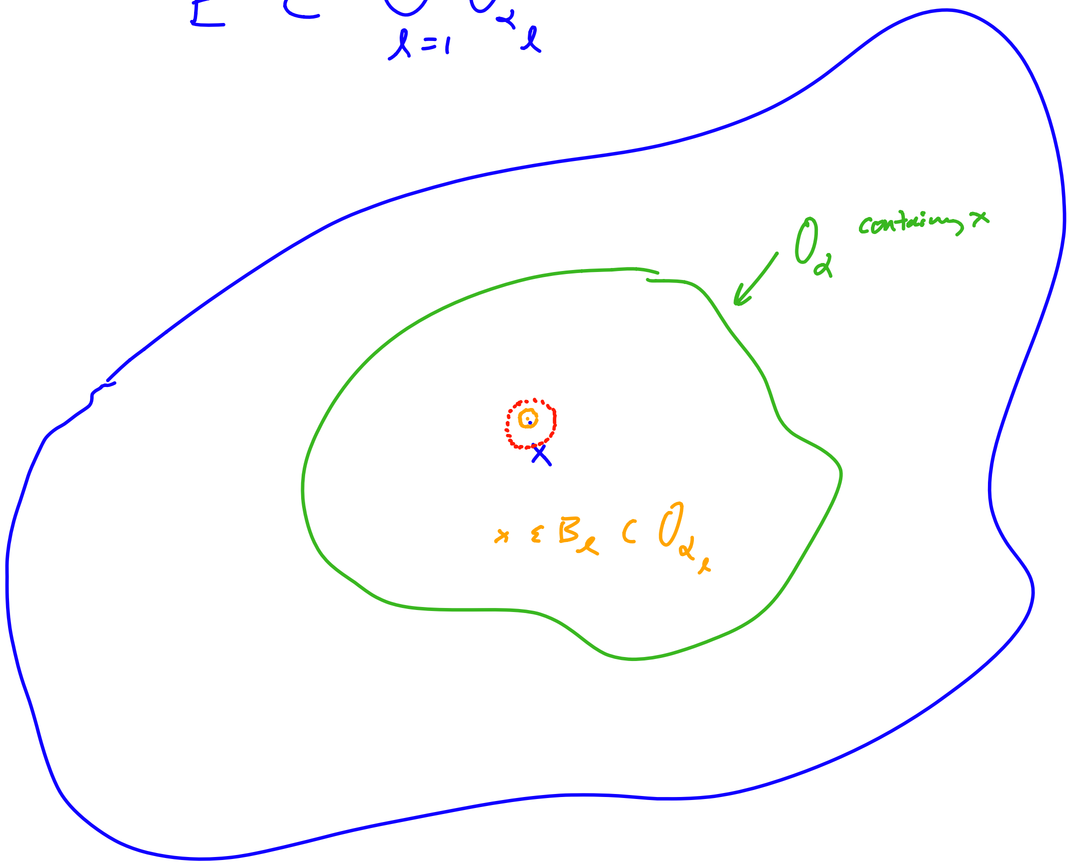


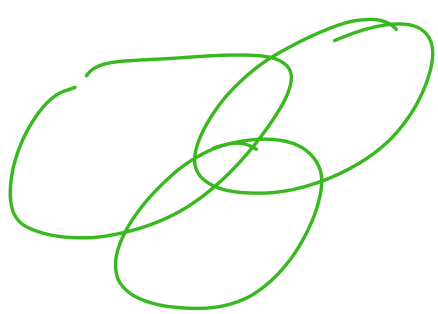

given $x \in E$

objective: find $B_\ell$ so $x \in B_\ell$ and $\exists$ $\alpha$ $\ni$ $B_\ell \subset O_\alpha$ and we will name this $\alpha$, $\alpha_\ell$

$\boxed{\text{objective : given } x \in E \text{ find } B_\ell \text{ and } O_{\alpha_\ell} \text{ so that } x \in B_\ell \subset O_{\alpha_\ell}}$

$$E \subset \bigcup_{\ell=1} \mathcal{O}_{\alpha_\ell}$$



① $\underline{X}$ separable $\Rightarrow$ $\exists$ $F = \{f_i\}_{i=1}^{\infty}$

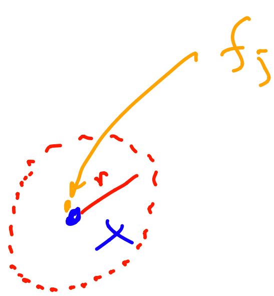
$\ni$ $\forall$ $x \in \underline{X}$ and any $\varepsilon > 0$ $\exists$ $j$

so that $\rho(f_j, x) < \varepsilon$

② $B_{j,k}$ = . .

Relabel $B_{j,k} \longrightarrow B_\ell$

$\rho(f_j, x) < \frac{1}{2^{k+1}}$




$\frac{1}{2^k} < r$

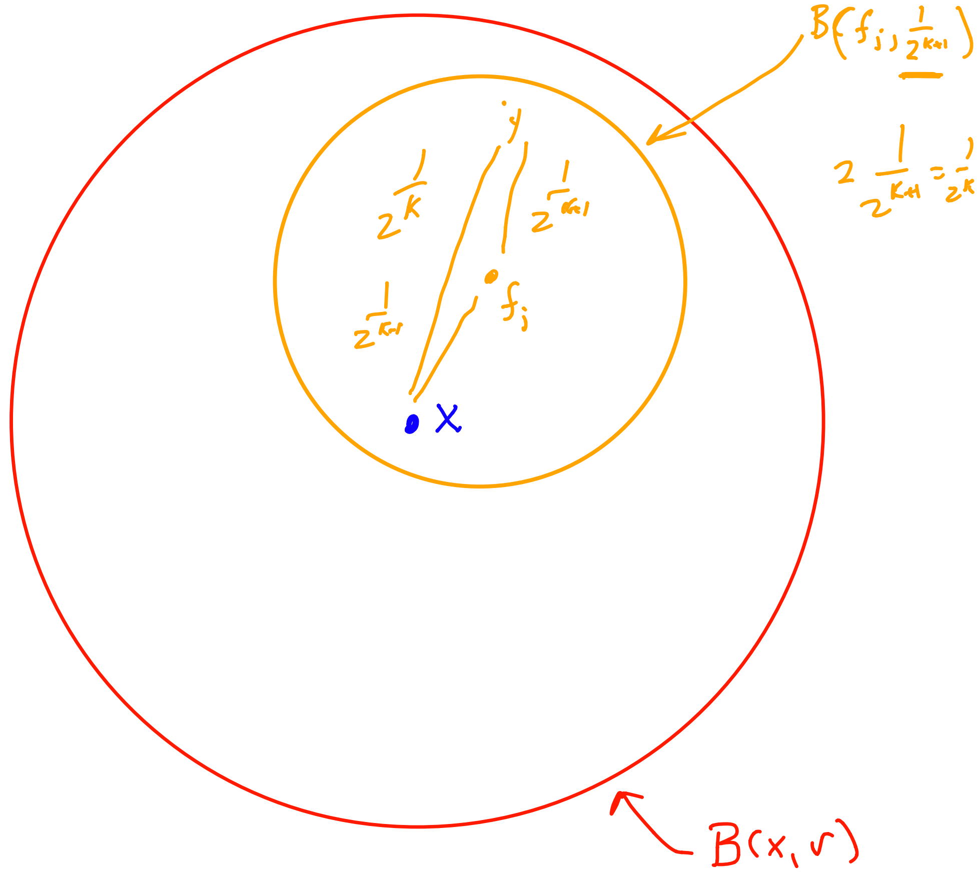
$\frac{1}{2^{k+1}}$

$\rho(f_j, x) < \frac{1}{2^{k+1}} \Rightarrow x \in B(f_j, \frac{1}{2^{k+1}}) \subset B(x, r)$

$\frac{1}{2^k}$

$\frac{1}{2^{k+1}}$

$\frac{1}{2^k} \cdot \frac{1}{2^1}$




$$B_{j,k+1} = B_\ell$$

$y, w \in B(x, r)$

$\rho(y, w) < 2r$