Proof that {Closed and Bounded} $\Rightarrow$ {Finite Subcovers}

Kevin R. Vixie

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This proof is both simple – once you have the geometric outline in your head – and complex – if you let the details hide the big picture from you.

Important Comments: You should not expect to read this quickly, nor should you expect to read it without having a paper and pencil handy to make sketches in your quest to understand and master the ideas here. There is nothing in these notes that is completely new to you – i.e. you have seen all the ideas in the book and lectures. But you most likely have not made these ideas part of your subconscious tool set yet, so the process is still going to take some time.

You should definitely try to master these notes before class. I will also go over these notes, in detail, with tons of pictures, in class, so if the struggle to master the notes overwhelms you, help is on the way!

1 The Proof in Detailed Steps

The idea is very close to what I said in class on Friday.

1. We start with a closed and bounded set $E \subset \mathbb{R}^n$.
2. We assume we have already shown that (a) $\mathbb{R}^n$ is complete – that is, that every Cauchy sequence converges to some point in the space and (b) every bounded sequence has a convergent subsequence.
3. Assume that $E \subset \bigcup_{i=1}^{\infty} U_i$ where each of the $U_i \subset \mathbb{R}^n$ are open.
4. **See the figure below for the next few steps ...**

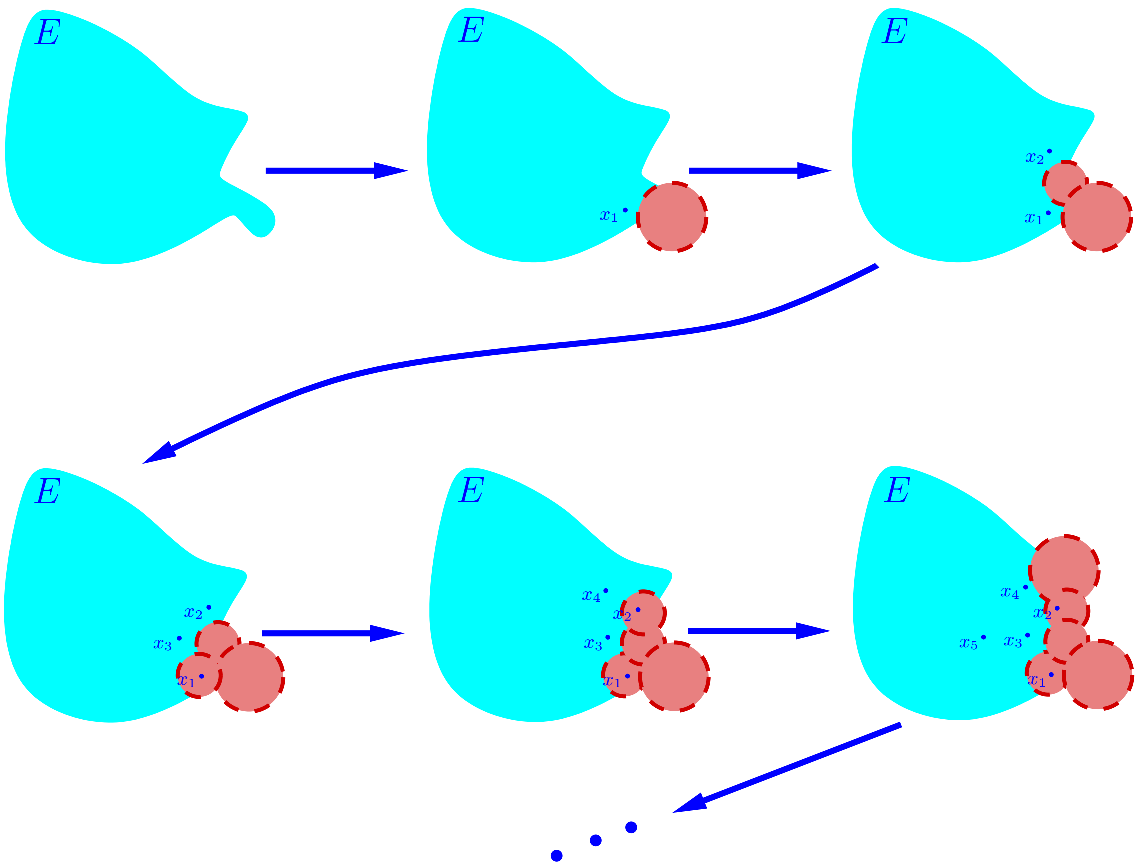

Figure 1: The process of choosing a sequence is simple ...

5. Now: choose $x_1 \in E \cap (U_1)^c$. I.e. x_1 is in E, but not in U_1.
6. Next choose $x_2 \in E \cap (U_1 \cup U_2)^c$. I.e. x_2 is in E, but not in $U_1 \cup U_2$.
7. ... and choose $x_3 \in E \cap (U_1 \cup U_2 \cup U_3)^c$. I.e. x_3 is in E, but not in $U_1 \cup U_2 \cup U_3$.
8. ... you see the pattern: $x_n \in E \cap (U_1 \cup U_2 \cup \cdots \cup U_n)^c$. I.e. x_n is in E, but not in $U_1 \cup U_2 \cup \cdots \cup U_n$.
9. Notice that if there is ever a point where you can't find an x_k for some k, then that means the first $k-1$ open sets cover E and we are done – we have found a finite subcover of E.
10. If we never fail to choose an x_k for $k = 1, 2, 3$ ..., then we have chosen an infinite sequence from E and we know that there is a subsequence $\{x_{i_n}\}_{n=1}^{\infty}$ which converges to a point x^* in E: $\{x_{i_n}\}_{n=1}^{\infty} \to x^*$.
11. Now we notice something: every U_k covers at most only the first $k-1$ points in the sequence: $\{x_i\}_{i=1}^{k-1}$.
12. Now because $x^* \in E$ and the union of all the U_i's cover E, then there is some U_m such that $x^* \in U_m$. Since U_m is open this implies that for some $N < \infty$, the tail of the subsequence is in U_m: $\{x_{i_n}\}_{n=N}^{\infty} \subset U_m$.
13. Now we can reach a contradiction, because from Step(11) we have that U_m covers at most $\{x_i\}_{i=1}^{m-1}$ but from Step(12) we have that U_m contains x_i with i arbitrarily big.
14. That contradiction implies that we were not able to always pick an x_i for all i and so we halted the procedure and found a finite subcover.
15. ... And we are done!

2 Even More ...

Note that we can even have an uncountable cover of E and find a finite subcover (when E is closed and bounded). This follows because, from any cover of any set in $\mathbb{R}^n$, we can choose a countable subcover. Here is the proof:

Synopsis: If we have a cover of E by the open sets: $E \subset \bigcup_{\alpha \in \mathcal{A}} U_\alpha$, we know there is a countable subcover: $\{U_i\}_{i=1}^{\infty} \subset \{U_\alpha\}_{\alpha \in \mathcal{A}}$ and $E \subset \bigcup_{i=1}^{\infty} U_i$. We know this because R^n contains a set that (a) is countably infinite and (b) whose closure is all of $\mathbb{R}^n$. Here are the details ...

1. Because the set of rational numbers $\mathbb{Q}$ is countable, we can use the standard argument to show that $\mathbb{Q}^n$ is countable. (a zigzag argument to show that the product of two countable sets is countable: now iterate.)

2. Note That $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$: given any point $x \in \mathbb{R}^n$, there is a sequence $\{q_i\}_{i=1}^{\infty} \subset \mathbb{Q}^n$ that converges to x.

3. Now since $\mathbb{Q}^n$ is countable, the set of open balls centered on points in $\mathbb{Q}^n$ that have rational radius is also countable.

4. For every point $e \in E$, there is an open set U_α that contains e and because U_α is open, there is an open ball centered on e with radius $r > 0$, $B(e, r) \subset U_\alpha$.

5. Now choose a point $q_e \in \mathbb{Q}^n$ such that $|q_e - e| < \frac{r}{3}$ and a rational number k_e such that $\frac{r}{3} < k_e < \frac{2r}{3}$.

6. Observe that $B(q_e, k_e)$ contains e and is in U_α.

7. The collection of all such balls, $\{B(q_e, k_e)\}_{e \in E}$, is countably infinite at most (because there are only a countable infinity of balls with rational radii and centers in $\mathbb{Q}^n$), and together, they contain all of E.

8. Now choose one set from the open cover that contains each $B(q_e, k_e)$. Together these sets cover E and are countably infinite at most.

9. Done.